Intuiting the Quadratic Formula

Rob S

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(Note: This note intentionally uses casual and imprecise mathematical language, to emphasize the fact that concepts are meant to be thought about intuitively instead of rigorously.)

Given a quadratic equation in one variable with real coefficients $ax^2 + bx + c = 0$, it's well-known that the solution for x is given by the equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When my students studied this equation and its application in solving quadratics, they were often told by their teachers to simply memorize the formula, often using a song or mnemonic.

Most of my students, like with many people, were visual learners. It was difficult for them to intuit the abstract concept of prime numbers, until they were handed a stack of eleven or seventeen blocks and told to make a perfect rectangle out of them. Almost all the students who where learning the quadratic formula had already been exposed to the graph of a parabola and how its intersection with the x-axis were connected to the zeroes of the parabola's associated quadratic equation. That seemed like a good place to start.

We began with the graph of a standard parabola, where there were a few important things for the student to notice. To start, a parabola is an even function, so for any point (x_0, y_0) on the locus, $(-x_0, y_0)$ was also on the locus. Another way to phrase this was that the parabola was symmetric about the y axis. Also, when $x = 0$, $y = 0$; therefore on a graph, the parabola intersected the origin.

This was the only point in the graph where $y = 0$, so finding zeroes at this point was trivial.

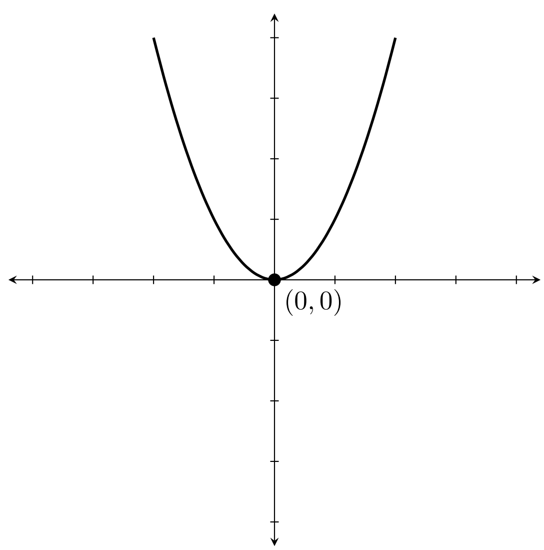

Figure 1: $y = x^2$

Modifying the equation to $y = ax^2$ and changing the value of a allowed us to scale the graph. The students understood this intuitively as stretching or squeezing the graph, and sometimes flipping the graph upside-down. But the center of the parabola did not move—it stayed right at the origin.

Right now the equation still only had one zero: $x = 0$. Simple enough.

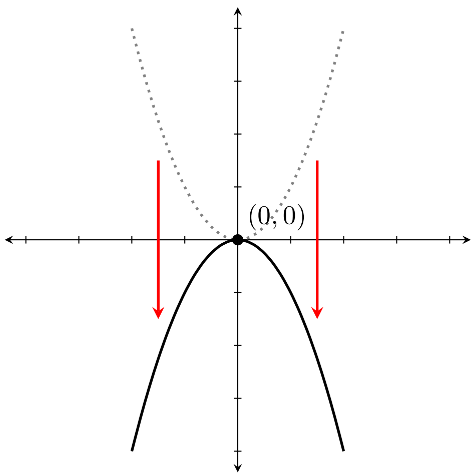

Figure 2: $y = -x^2$

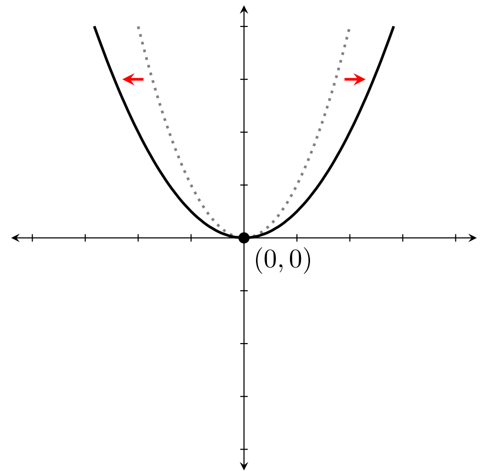

Figure 3: $y = \frac{x^2}{2}$

We then examined what happened to the parabola when a linear term was added to the equation, giving $y = ax^2 + bx$. One important thing remained true: when $x = 0$, $y = 0$. When plotting the locus, one "arm" of the parabola remained on the origin, and the parabola "slid" along the origin to the left or right depending on the value of b. Even if the parabola opened downward, some point on the parabola still stayed on the origin. Thus the effect of b was a translation or slide.

But now we had a second point where the parabola crossed the x axis, or equivalently where $y = 0$. Where was that second point? Factoring gave us a symbolic answer: $x = \frac{-b}{a}$, but we could even intuit this from the graph of the parabola. For example, when our equation was $y = x^2 + 3x$, we could see that our second zero was located at $x = -3$, and when we set the equation to $y = 2x^2 - 5x$, our second zero slid to $x = \frac{5}{2}$.

There was one more point to pay attention to: the center of the parabola.
Since the parabola was symmetric, the center of the parabola was halfway between each of the zeroes. If one zero was at $x = 0$ and the other was at $x = \frac{-b}{a}$, halfway between them would be $x = \frac{-b}{2a}$. This was, of course, part of the quadratic formula; most students recognized this right away. To find the y coordinate, all we had to do was plug our x value back into the equation, yielding $y = \frac{-b^2}{4a}$.

We now had a solid baseline for understanding the zeroes of any parabola without a constant term. Good stuff.

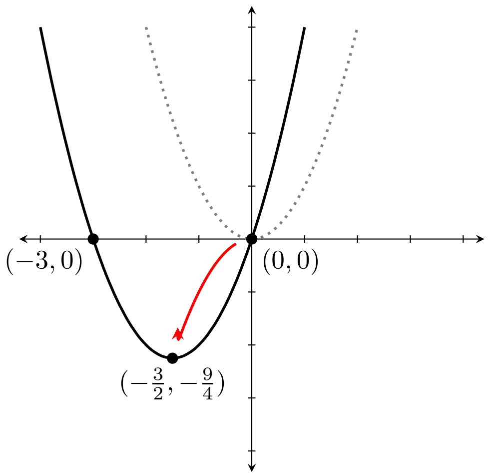

Figure 4: $y = x^2 + 3x$

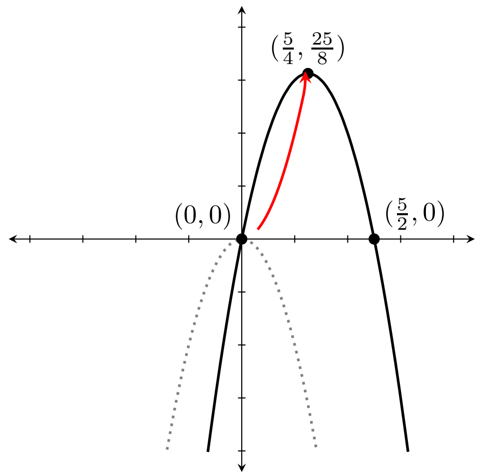

Figure 5: $y = -2x^2 + 5x$

Now it was time to introduce our last term: c. How could we deal with that?

The students knew from graphing other equations that constant terms caused the graph to shift up and down, or translate. So we added a few constants to see what happened. As we would have expected, adding a positive constant shifted the graph up, and a negative term shifted the graph down.

But how did our zeroes move? This depended on the orientation of the parabola, not just on the value of c. When the parabola opened upward ($a > 0$), moving the graph upward pushed the zeroes closer together, since the bottom of the parabola was narrower. When the parabola opened downward, however ($a < 0$), moving the graph upward pushed the zeroes farther apart, since the bottom of the parabola was now wider. The opposite of course applied when the graphs were moved downward.

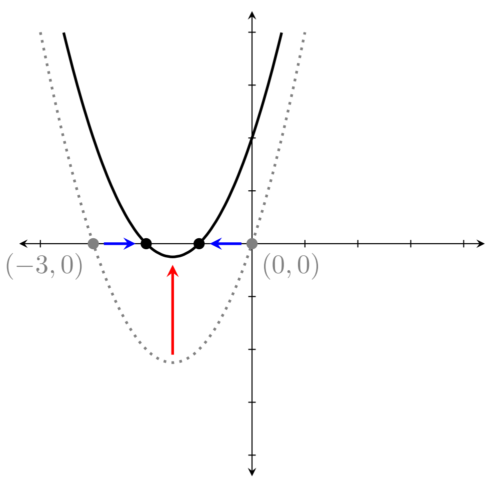

Figure 6: $y = x^2 + 3x + 2$

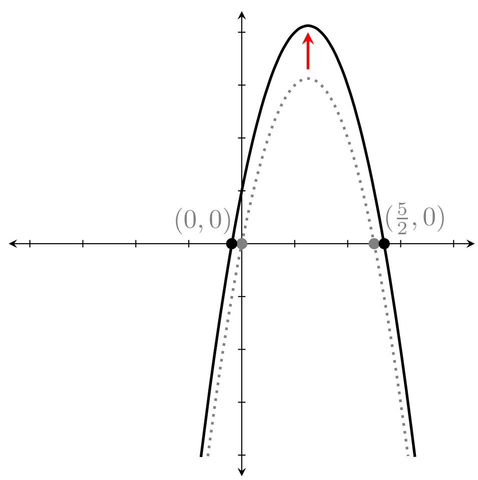

Figure 7: $y = -2x^2 + 5x + 1$

The final piece of the puzzle was to figure out how much the zeroes move by. But to do that, we needed to focus first on our center.

The center of the parabola, which we'd earlier found was at $(\frac{b}{-2a}, \frac{-b^2}{4a})$, had now been moved up by c. Our new y coordinate was thus $y = \frac{-b^2+4ac}{4a}$. That value represented the distance from the center to the x axis, and the absolute value of that distance was $\frac{b^2-4ac}{4a}$. At this point the students usually recognized the numerator as also being part of the quadratic formula. The pieces were starting to come together.

It was here that we went back to the effect of a, which was the only coefficient that scaled the parabola; b and c only served to move the graph around. How did scaling affect distances?
With a standard parabola, if $y = x^2$, then $x = \pm\sqrt{y}$. This told us that if we move some amount in the y direction, we had to move the square root of that in either x direction. But when we added a scaling factor $y = ax^2$, now $x = \pm\sqrt{\frac{y}{a}}$. That meant that our x movement had to be divided by a factor of $\sqrt{a}$.

We had noted earlier that the distance from the center of the parabola to the x axis was $\frac{b^2-4ac}{4a}$. To figure out how far the zeroes (on the x axis) were from the center, we needed to plug them into our scaled equation above. That gave us $x = \pm\frac{\sqrt{b^2-4ac}}{2a}$. The lights were coming on in the students' minds now.

The final step was to note that this was merely the distance of the zeroes from the center line of the parabola. The center line, as we had found earlier, was $x = \frac{-b}{2a}$, so we needed to add these two results together. And our final result was $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

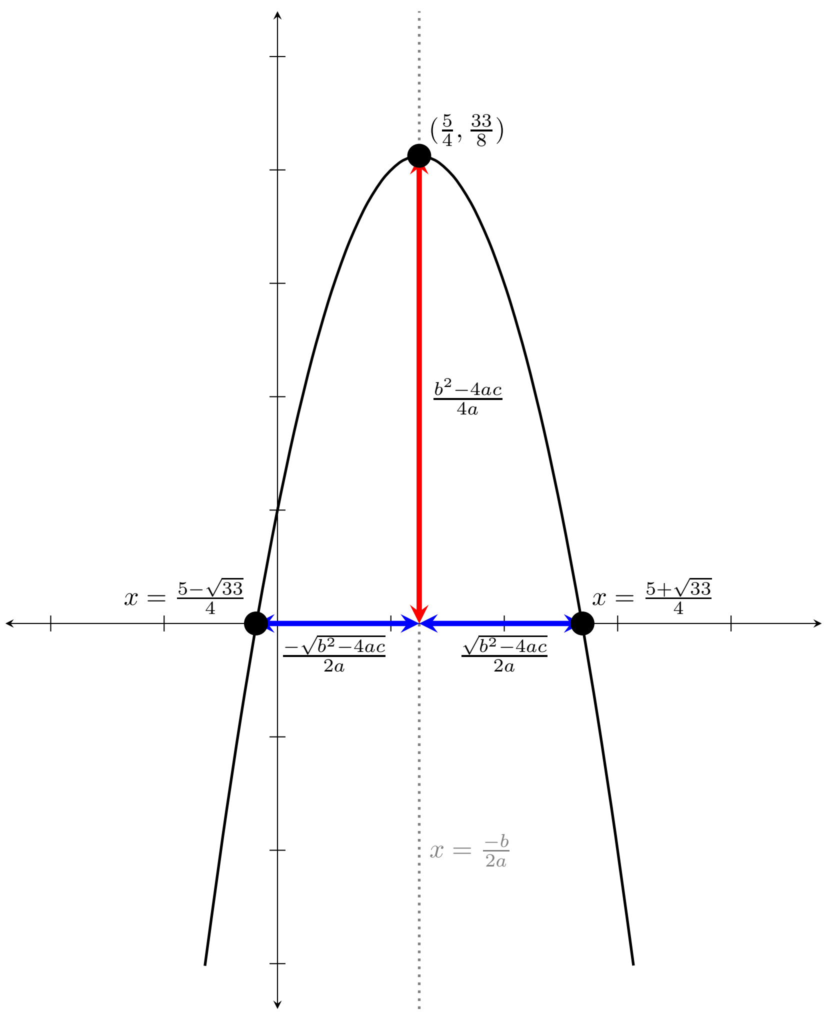

Figure 8: $y = -2x^2 + 5x + 1$, with relative distances to zeroes

The overall process was to see how the graph of a parabola moved when coefficients were added to it, determine where the center of the parabola wound up after that movement, and figure out how far the zeroes were from that center. This allowed us to build up each major piece of the quadratic formula and see where it came from. While the exact placement of each coefficient in the formula could only be determined from calculation, the process at least helped the student understand why each major component was there.

I've been surprised how many of my students understood the quadratic formula much more clearly after this exercise. You may find that it helps a student you know (or even yourself!) to get a better grasp on the formula.

[Last updated: 2021-11-19]