Computing the Character Table of a 2-local Maximal Subgroup of the Monster

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Outline

Aim

Compute the character table of $2^{5+10+20} \cdot (S_3 \times L_5(2))$.

Motivation

Character tables nicely "summarise" groups.

All groups factor into a composition series of simple groups, so their character tables are of particular interest.

Whole books have been published on the subject (e.g. the ATLAS)!

Simple groups still have a subgroup structure: look at their *maximal* subgroups too.

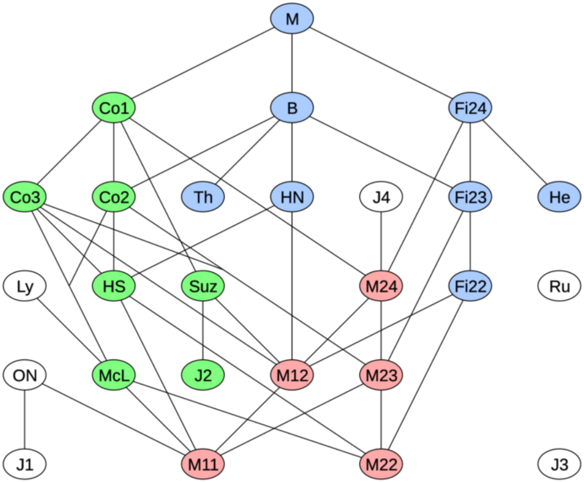
Sporadic groups & the Monster

The **CFSG** (Classification of Finite Simple Groups) asserts all simple groups, except 26 **sporadic groups**, belong to one of 18 infinite families.

The Monster $\mathbb{M}$ is the largest sporadic group by far: $\approx 2 \times 10^{20}$ times bigger than the Baby Monster $\mathbb{B}$ in second place. First constructed by Griess in 1982 as a 196884-dimensional matrix group.

$$|\mathbb{M}| \approx 8 \times 10^{53}.$$

Sporadic groups & the Monster



The Sporadic Groups (SporadicGroups.png, Wikimedia Commons: CC BY-SA 4.0)

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Character tables of the sporadic simple groups have been known for over 4 decades.

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Tables for their maximal subgroups have taken longer: for 25 of the sporadic groups, these were completed by mid-2012.

But the problems for the Monster's maximal subgroups remained: even the list of subgroups wasn't settled until 2024.

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As a permutation group: $\approx 10^{20}$ points (vs. $\mathbb{B}$'s $\approx 10^{10}$)

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Subgroups may be more amenable, but determining their precise structure is itself a challenge.

Bray and Wilson (2006) give permutation representations for all then-known maximal subgroups of $\mathbb{M}$, **except** the 2-local cases.

Sophisticated computational methods of Holmes and Wilson allowed most maximal subgroups of $\mathbb{M}$ to be determined by 2013.

mmgroup

Unfortunately, Holmes and Wilson's code is unreleased and still undesirably slow. Only in 2022 did fast group operations in $\mathbb{M}$ become possible: Seysen's `mmgroup` Python package.

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except $2^{5+10+20}{\cdot}(\mathrm{S}_3 \times \mathrm{L}_5(2))$.

The Maximal Subgroups of $\mathbb{M}$

$2{\cdot}\mathbb{B}$	$(D_{10} \times HN){\cdot}2$	$(A_5 \times A_{12}){:}2$
$2^{1+24}{\cdot}Co_1$	$5^{1+6}{:}2{\cdot}J_2{:}4$	$(A_6 \times A_6 \times A_6).(2 \times S_4)$
$2^{2.2}E_6(2){:}S_3$	$(5^2{:}4{\cdot}2^2 \times U_3(5)){:}S_3$	$(A_5 \times U_3(8){:}3){:}2$
$2^{2+11+22}.(M_{24} \times S_3)$	$5^{2+2+4}{:}(S_3 \times GL_2(5))$	$(PSL_3(2) \times S_4(4){:}2){\cdot}2$
$2^{3+6+12+18}.(PSL_3(2) \times 3.S_6)$	$5^{3+3}.(2 \times PSL_3(5))$	$(PSL_2(11) \times M_{12}){:}2$
$2^{5+10+20}.(S_3 \times PSL_5(2))$	$5^4{:}(3 \times 2{\cdot}PSL_2(25)){:}2$	$(A_7 \times (A_5 \times A_5){:}2^2){:}2$
$2^{10+16}.O_{10}^{+}(2)$	$(7{:}3 \times He){:}2$	$M_{11} \times A_6{\cdot}2^2$
$3{\cdot}Fi_{24}$	$7^{1+4}{:}(3 \times 2.S_7)$	$(S_5 \times S_5 \times S_5){:}S_3$
$3^{1+12}{\cdot}2{\cdot}Suz{:}2$	$(7^2{:}(3 \times 2.A_4) \times PSL_2(7)){:}2$	$(PSL_2(11) \times PSL_2(11)){:}4$
$S_3 \times Th$	$7^{2+1+2}{:}GL_2(7)$	$U_3(4){:}4$
$(3^2{:}2 \times O_8^{+}(3)){\cdot}S_4$	$7^2{:}SL_2(7)$	$PSL_2(71)$
$3^{2+5+10}{:}(M_{11} \times 2.S_4)$	$11^2{:}(5{\times}2.A_5)$	~~$PSL_2(59)$~~ 59:29
$3^{3+2+6+6}{:}(PSL_3(3) \times SD_{16})$	$(13{:}6 \times PSL_3(3)){\cdot}2$	$PSL_2(41)$
$3^8{\cdot}O_8^{-}(3).2$	$13^{1+2}{:}(3 \times 4.S_4)$	$PGL_2(29)$
	$13^2{:}SL_2(13){:}4$	$PGL_2(19)$
	$41{:}40$	$PGL_2(13)$

The maximal subgroups of $\mathbb{M}$.

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The Choice of Tools

Most character tables of maximal subgroups of $\mathbb{M}$ have been computed in Magma with Bray and Wilson's permutation representations.

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Problem: permutation representations of 2-local subgroups seem elusive.

Could use `mmgroup` representations, but `mmgroup` currently has limited functionality.

Hybrid Groups

Hulpke (2024) handles the 2-local subgroup $2^{10+16} \cdot \mathrm{O}_{10}^{+}(2)$ differently: as a *hybrid group* in GAP.

The implementation, based on Dietrich and Hulpke (2021), represents a group G by a solvable normal subgroup H, the quotient G/H, and the conjugation action of G/H on H.

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This works for $2^{5+10+20} \cdot (S_3 \times L_5(2))$ too: we can obtain the hybrid representation from `mmgroup`, and even translate between it and the `mmgroup` copy.

The Computation (I)

Mostly straightforward: use GAP commands.

For computing characters, adopt the more efficient approach used in Magma, which uses Brauer's Theorem on Induced Characters:

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Implementing this in GAP specifically for $2^{5+10+20} \cdot (\mathrm{S}_3 \times \mathrm{L}_5(2))$ is mostly OK, especially since elementary subgroups are so nice.

The Computation (II)

Problem: we need the character table of $P \in \mathrm{Syl}_2(2^{5+10+20}{\cdot}(\mathrm{S}_3 \times \mathrm{L}_5(2)))$, which is huge: $|P| = 2^{46}$, and P's character table is 26758×26758.

Introduce some optimisations.

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Thank you!