The Maximal Subgroups of the Monster Group

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Outline

Aim

Construct instances of each of the 46 conjugacy classes of maximal subgroups in the Monster in `mmgroup`.

Motivation

To understand things, try decomposing them.

Repeatedly factoring maximal normal subgroups out of a group gives its **composition series**. The Jordan–Hölder theorem shows this is a well-defined factorisation into simple groups.

Simple groups still have a subgroup structure: look at their *maximal* subgroups.

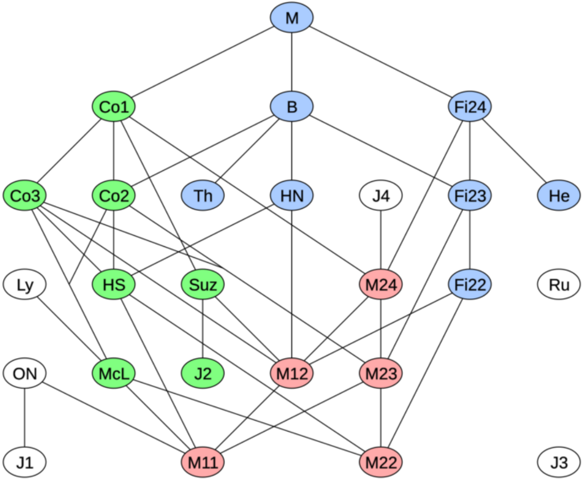
Sporadic groups & the Monster

The **CFSG** (Classification of Finite Simple Groups) asserts all simple groups, except 26 **sporadic groups**, belong to one of 18 infinite families.

The Monster $\mathbb{M}$ is the largest sporadic group by far: $\approx 2 \times 10^{20}$ times bigger than the Baby Monster $\mathbb{B}$ in second place. First constructed by Griess in 1982 as a 196884-dimensional matrix group.

$$|\mathbb{M}| \approx 8 \times 10^{53}.$$

Sporadic groups & the Monster



The Sporadic Groups (SporadicGroups.png, Wikimedia Commons: CC BY-SA 4.0)

A Monster of a Problem

The maximal subgroups for 25 sporadic groups were determined by 1999, with increasing computational demands for larger groups.

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As a permutation group: $\approx 10^{20}$ points (vs. $\mathbb{B}$'s $\approx 10^{10}$)

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Sophisticated computational methods of Holmes and Wilson allowed most potential subgroups of $\mathbb{M}$ to be handled by 2013.

mmgroup

Unfortunately, Holmes and Wilson's code is unreleased and still undesirably slow. Only in 2022 did fast group operations in $\mathbb{M}$ become possible: Seysen's `mmgroup` Python package.

Dietrich, Lee, and Popiel used `mmgroup` to complete the classification of maximal subgroups of $\mathbb{M}$ in 2023.

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The Maximal Subgroups of $\mathbb{M}$

$2{\cdot}\mathbb{B}$	$(D_{10} \times HN){\cdot}2$	$(A_5 \times A_{12}){:}2$
$2^{1+24}{\cdot}Co_1$	$5^{1+6}{:}2{\cdot}J_2{:}4$	$(A_6 \times A_6 \times A_6).(2 \times S_4)$
$2^{2.2}E_6(2){:}S_3$	$\left(5^2{:}4{\cdot}2^2 \times U_3(5)\right){:}S_3$	$(A_5 \times U_3(8){:}3){:}2$
$2^{2+11+22}.(M_{24} \times S_3)$	$5^{2+2+4}{:}(S_3 \times GL_2(5))$	$(PSL_3(2) \times S_4(4){:}2){\cdot}2$
$2^{3+6+12+18}.(PSL_3(2) \times 3.S_6)$	$5^{3+3}.(2 \times PSL_3(5))$	$(PSL_2(11) \times M_{12}){:}2$
$2^{5+10+20}.(S_3 \times PSL_5(2))$	$5^4{:}(3 \times 2{\cdot}PSL_2(25)){:}2$	$\left(A_7 \times (A_5 \times A_5){:}2^2\right){:}2$
$2^{10+16}{\cdot}O_{10}^+(2)$	$(7{:}3 \times He){:}2$	$M_{11} \times A_6{\cdot}2^2$
$3{\cdot}Fi_{24}$	$7^{1+4}{:}(3 \times 2.S_7)$	$(S_5 \times S_5 \times S_5){:}S_3$
$3^{1+12}{\cdot}2{\cdot}Suz{:}2$	$\left(7^2{:}(3 \times 2.A_4) \times PSL_2(7)\right){:}2$	$(PSL_2(11) \times PSL_2(11)){:}4$
$S_3 \times Th$	$7^{2+1+2}{:}GL_2(7)$	$U_3(4){:}4$
$\left(3^2{:}2 \times O_8^+(3)\right){\cdot}S_4$	$7^2{:}SL_2(7)$	$PSL_2(71)$
$3^{2+5+10}{:}(M_{11} \times 2.S_4)$	$11^2{:}(5 \times 2.A_5)$	~~$PSL_2(59)$~~
$3^{3+2+6+6}{:}(PSL_3(3) \times SD_{16})$	$(13{:}6 \times PSL_3(3)){\cdot}2$	$PSL_2(41)$
$3^8{\cdot}O_8^-(3).2$	$13^{1+2}{:}(3 \times 4.S_4)$	$PGL_2(29)$
	$13^2{:}SL_2(13){:}4$	$PGL_2(19)$
$59{:}29$	$41{:}40$	$PGL_2(13)$

The maximal subgroups of $\mathbb{M}$.

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3. Find elements of H in another subgroup already known.
4. Repeat until the elements found generate H.

2B centralisers

We need some subgroups of $\mathbb{M}$ to start.

The Monster contains a conjugacy class of involutions 2B. Centralisers $C_{\mathbb{M}}(x)$ of $x \in$ 2B are very convenient:

- the centraliser $\mathbf{G} = C_{\mathbb{M}}(z)$ of a certain $z \in$ 2B is built into `mmgroup`.
- for any $x \in$ 2B, `mmgroup` can efficiently find h such that $z = x^h$, i.e. $C_{\mathbb{M}}(x)^h = \mathbf{G}$.
- many subgroups contain a 2B involution.

2B centralisers

We need some subgroups of $\mathbb{M}$ to start.

The Monster contains a conjugacy class of involutions 2B. Centralisers $C_{\mathbb{M}}(x)$ of $x \in 2B$ are very convenient:

- computations in **G** are extremely fast.
- `mmgroup` can efficiently compute some characters at any $x \in \mathbf{G}$.
- `mmgroup` provides a homomorphism $\phi : \mathbf{G} \to Co_1 < GL_{24}(2)$ which allows us to work in GAP / Magma.

Projection and Lifting

Key Point

`mmgroup` provides a homomorphism $\phi : \mathbf{G} \to \mathrm{Co}_1 < \mathrm{GL}_{24}(2)$ which allows us to work in GAP / Magma.

The centraliser or normaliser of $x \in \mathbf{G}$ in $\phi(\mathbf{G})$ contains its centraliser or normaliser in $\mathbf{G}$. We can generate the supergroups in GAP / Magma.

Fix generators a, b of $\mathbf{G}$ and write any u in the supergroups as a word in $\phi(a), \phi(b)$. Evaluating such a word gives $x \in \phi^{-1}(u)$.

Centralising or normalising elements in $x \ker \phi$ are found using linear algebra: conjugation by $\mathbf{G}$ induces automorphisms of $\ker \phi \cong 2^{1+24}$.

Standard generators

Sometimes, we search in subgroups $G \neq \mathbf{G}$. Finding elements of a group H in such G is harder.

Fortunately, if the generators of G are **standard generators** — satisfy a certain presentations for G — words for elements that generate maximal subgroups of G are often known.

These words can be found online alongside efficient algorithms for finding standard generators.

A Case Study

Consider $H \cong \mathrm{PSL}_2(59) < \mathbb{M}$. The structure of H means $H = \langle \mathrm{A}_5, i \rangle$, where i has order 2 and centralises $\mathrm{D}_{10} < \mathrm{A}_5$.

We already have $G_1 = \mathrm{A}_5$. Consider $C_{\mathbb{M}}(A)$ for some $A \cong \mathrm{D}_{10} < G_1$. All elements of order 2 in G_1 are of class 2B, so we can work in **G** to find $C_{\mathbb{M}}(A)$ and the potential 2B elements i.

Testing reveals which candidates extend $G_1 \cong \mathrm{A}_5$ to $\mathrm{PSL}_2(59)$.

A Case Study (Take 2)

Since $\mathrm{PSL}_2(59)$ is not a subgroup of $\mathbb{M}$, there is a maximal subgroup $H \cong 59{:}29$ instead.

This is the normaliser of an element of order 59 in the Monster.

The problem: what known subgroups can we search in?

Remember:

We need already-constructed subgroups of $\mathbb{M}$ which intersect H in distinct subgroups.

Thank you!

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